

DISCRETE DYNAMICS IN IMPLICIT FORM

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ABSTRACT. A notion of implicit difference equation on a Lie groupoid is introduced and an algorithm for extracting the integrable part (backward or/and forward) is formulated. As an application, we prove that discrete Lagrangian dynamics on a Lie groupoid G may be described in terms of Lagrangian implicit difference equations of the corresponding cotangent groupoid T^*G . Other situations include finite difference methods for time-dependent linear differential-algebraic equations and discrete nonholonomic Lagrangian systems, as particular examples.

Dedicated to Ernesto Lacomba on the occasion of his 65th birthday

1. INTRODUCTION

Lie algebroids and groupoids have deserved a lot of interest in recent years since these concepts generalize the traditional framework of tangent bundles and its discrete version, cartesian product of manifolds, to more general situations. In particular, it is well-known that the geometric description of the Euler-Lagrange equations of a mechanical system determined by a Lagrangian function L , relies on the intrinsic geometry of the tangent bundle TQ , the velocity phase space of a configuration manifold Q . In the case when the Lagrangian L is invariant under the action of a Lie group G , the description, in this case, relies on the geometry of the quotient space TQ/G and the equations describing the dynamics are called Lagrange-Poincaré equations [4]. In this sense, Weinstein [29] showed that the common geometric structure of the Lagrange-Poincaré equations is essentially the same as the one of the Euler-Lagrange equations, namely that of a Lie algebroid. In the case of Lagrangian systems on the usual tangent bundles, the tangent bundle carries a canonical Lie algebroid structure which is given by the usual Lie algebra of vector fields on Q . In the case of reduced lagrangian systems, we use the Atiyah algebroid $TQ/G \rightarrow Q/G$ to describe the evolution equations.

In [18], it is described geometrically discrete Lagrangian and Hamiltonian Mechanics on Lie groupoids, in particular, the type of equations analyzed include the classical discrete Euler-Lagrange equations, the discrete Euler-Poincaré and discrete Lagrange-Poincaré equations. These results have applications for the construction of geometric integrators for continuous Lagrangian systems (reduced or not) [9, 24].

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On the other hand, in [26, 27], W.M. Tulczyjew proved that it is possible to interpret the ordinary Lagrangian and Hamiltonian dynamics as Lagrangian submanifolds of suitable special symplectic manifolds. These results were successfully extended to general Lie algebroids in [12]. More generally, in [14, 15, 16] implicit differential equations are considered. More precisely, an implicit differential equation on a manifold Q is a submanifold $D \subset TQ$. Given such a datum, the problem of integrability is discussed, looking for a subset $S \subseteq D$ where, for any $v \in S$ there exists a curve $\gamma : I \rightarrow Q$ such that $\dot{\gamma}(0) = v$ and $\dot{\gamma}(t) \in D$ for any $t \in I$. An algorithm for extracting the integrable part of an implicit differential equation is formulated. More precisely, if $D \subset TQ$ is an implicit differential equation, the following sequence $\{(C^k, D^k)\}$ is considered

$$\begin{aligned} D^0 &= D, & C^0 &= \tau_Q(D), \\ D^k &= D^{k-1} \cap TC^{k-1}, & C^k &= \tau_Q(D^k), \end{aligned}$$

where $\tau_Q : TQ \rightarrow Q$ is the canonical projection. Clearly, $D^k \subseteq D^{k-1}$ and $C^{k-1} \subseteq C^k$ for any k . Under some smoothness assumptions, it is proved in [15] that if the algorithm stabilizes, that is, there exists \bar{k} such that $D^{\bar{k}} = D^{\bar{k}-1}$, then $D^{\bar{k}} \subseteq D$ is the (possibly empty) integrable part of D . The relation with Lagrangian mechanics is the following one. Given a Lagrangian $L : TP \rightarrow \mathbb{R}$, one can consider the implicit differential equation $S_L \subseteq T(T^*P)$ using the image of the differential of L , $dL(TP) \subset T^*(TP)$ and the canonical isomorphism $A_M : T^*(TP) \rightarrow T(T^*P)$. If we apply the integrability algorithm, the first step will give us the solution of the Euler-Lagrange equations. Indeed, if (q^i, v^i) are local coordinates on TP then

$$\begin{aligned} S_L &= \{(q^i, \frac{\partial L}{\partial v^i}; v^i, \frac{\partial L}{\partial q^i})\}, \\ (S_L)^1 &= \{(q^i, \frac{\partial L}{\partial v^i}; v^i, \frac{\partial L}{\partial q^i}) \mid \dot{q}^i = v^i, \quad \frac{d}{dt} \left(\frac{\partial L}{\partial v^i} \right) = \frac{\partial L}{\partial q^i} \}. \end{aligned}$$

Taking as starting point the results of discrete mechanics on Lie groupoids [10, 18] and a remark on implicit differential equations on Lie algebroids in [11], in this paper, submanifolds of a Lie groupoid G are interpreted as systems of implicit difference equations on G . We study the problem of integrability of these implicit difference equations, proposing an algorithm which extracts the integrable part (backward and/or forward) of this type of systems and analyze the geometric properties of them. In our opinion, this approach offers greater conceptual clarity. Discrete Lagrangian systems (unconstrained and nonholonomic) are studied within the framework of implicit difference equations. Moreover, for the unconstrained case, the resulting implicit difference equation is a Lagrangian submanifold, thus obtaining a discrete version of the results in [12]. Other examples include finite difference methods for time-dependent linear differential algebraic equations. The results about implicit difference equations are useful for systems more general than the studied on this paper; in particular, discrete optimal control theories, discrete systems with external constraints ... See for instance [10, 20].

2. PRELIMINARIES: LIE ALGEBROIDS AND LIE GROUPOIDS

In this Section, we will recall the definition of a Lie groupoid and some generalities about them are explained (for more details, see [13]).

A **groupoid** over a set M is a set G together with the following structural maps:

- A pair of surjective maps $\alpha : G \rightarrow M$, the **source**, and $\beta : G \rightarrow M$, the **target**. These maps define the set of composable pairs

$$G_2 = \{(g, h) \in G \times G / \beta(g) = \alpha(h)\}.$$

- A **multiplication** $m : G_2 \rightarrow G$, to be denoted simply by $m(g, h) = gh$, such that
 - $\alpha(gh) = \alpha(g)$ and $\beta(gh) = \beta(h)$.
 - $g(hk) = (gh)k$.
- An **identity section** $\epsilon : M \rightarrow G$ such that
 - $\epsilon(\alpha(g))g = g$ and $g\epsilon(\beta(g)) = g$.
- An **inversion map** $i : G \rightarrow G$, to be denoted simply by $i(g) = g^{-1}$, such that
 - $g^{-1}g = \epsilon(\beta(g))$ and $gg^{-1} = \epsilon(\alpha(g))$.

A groupoid G over a set M will be denoted simply by the symbol $G \rightrightarrows M$.

The groupoid $G \rightrightarrows M$ is said to be a **Lie groupoid** if G and M are manifolds and all the structural maps are differentiable with α and β differentiable submersions. If $G \rightrightarrows M$ is a Lie groupoid then m is a submersion, ϵ is an injective immersion and i is a diffeomorphism. Moreover, if $x \in M$, $\alpha^{-1}(x)$ (resp., $\beta^{-1}(x)$) will be said the **α -fiber** (resp., the **β -fiber**) of x .

On the other hand, if $g \in G$ then the **left-translation by $g \in G$** and the **right-translation by g** are the diffeomorphisms

$$\begin{aligned} l_g : \alpha^{-1}(\beta(g)) &\longrightarrow \alpha^{-1}(\alpha(g)) & ; & \quad h \longrightarrow l_g(h) = gh, \\ r_g : \beta^{-1}(\alpha(g)) &\longrightarrow \beta^{-1}(\beta(g)) & ; & \quad h \longrightarrow r_g(h) = hg. \end{aligned}$$

Note that $l_g^{-1} = l_{g^{-1}}$ and $r_g^{-1} = r_{g^{-1}}$.

A vector field \tilde{X} on G is said to be **left-invariant** (resp., **right-invariant**) if it is tangent to the fibers of α (resp., β) and $\tilde{X}(gh) = (T_h l_g)(\tilde{X}_h)$ (resp., $\tilde{X}(gh) = (T_g r_h)(\tilde{X}_g)$), for $(g, h) \in G_2$.

Now, we will recall the definition of the **Lie algebroid associated with G** .

We consider the vector bundle $\tau : AG \rightarrow M$, whose fiber at a point $x \in M$ is $A_x G = V_{\epsilon(x)} \alpha = \text{Ker}(T_{\epsilon(x)} \alpha)$. It is easy to prove that there exists a bijection between the space of sections of τ , $\Gamma(\tau)$, and the set of left-invariant (resp., right-invariant) vector fields on G . If X is a section of $\tau : AG \rightarrow M$, the corresponding left-invariant (resp., right-invariant) vector field on G will be denoted \overleftarrow{X} (resp., \overrightarrow{X}), where

$$\overleftarrow{X}(g) = (T_{\epsilon(\beta(g))} l_g)(X(\beta(g))), \quad (2.1)$$

$$\overrightarrow{X}(g) = -(T_{\epsilon(\alpha(g))} r_g)((T_{\epsilon(\alpha(g))} i)(X(\alpha(g)))), \quad (2.2)$$

for $g \in G$.

Using the above facts, we may introduce a Lie algebroid structure $(\llbracket \cdot, \cdot \rrbracket, \rho)$ on AG , which is defined by

$$\overleftarrow{\llbracket X, Y \rrbracket} = [\overleftarrow{X}, \overleftarrow{Y}], \quad \rho(X)(x) = (T_{\epsilon(x)}\beta)(X(x)), \quad (2.3)$$

for $X, Y \in \Gamma(\tau)$ and $x \in M$. We recall that a **Lie algebroid** A over a manifold M is a real vector bundle $\tau : A \rightarrow M$ together with a Lie bracket $\llbracket \cdot, \cdot \rrbracket$ on the space $\Gamma(\tau)$ of the global cross sections of $\tau : A \rightarrow M$ and a bundle map, called **the anchor map**, such that if we also denote by $\rho : \Gamma(\tau) \rightarrow \mathfrak{X}(M)$ the homomorphism of $C^\infty(M)$ -modules induced by the anchor map then

$$\llbracket X, fY \rrbracket = f\llbracket X, Y \rrbracket + \rho(X)(f)Y,$$

for $X, Y \in \Gamma(\tau)$ and $f \in C^\infty(M)$. The triple $(A, \llbracket \cdot, \cdot \rrbracket, \rho)$ is called a Lie algebroid over M (see [13]).

Next, we will present some examples of Lie groupoids and their associated Lie algebroids which will be useful for our purposes.

Any Lie group G is a Lie groupoid over $\{\mathfrak{e}\}$, the identity element of G , and the Lie algebroid associated with G is just the Lie algebra \mathfrak{g} of G . On the other hand, given a manifold M , the product manifold $M \times M$ is a Lie groupoid over M , called the **pair or banal groupoid**, in the following way: α is the projection onto the first factor and β is the projection onto the second factor, $m((x, y), (y, z)) = (x, z)$, for $(x, y), (y, z) \in (M \times M)_2$, $\epsilon(x) = (x, x)$, for all $x \in M$, and $i(x, y) = (y, x)$. The Lie algebroid $A(M \times M)$ of this groupoid is isomorphic to the tangent bundle $\tau_M : TM \rightarrow M$.

The cotangent groupoid. (See [5], for details). Let $G \rightrightarrows M$ be a Lie groupoid. If A^*G is the dual bundle to AG then the cotangent bundle T^*G is a Lie groupoid over A^*G . The projections $\tilde{\beta}$ and $\tilde{\alpha}$, the partial multiplication \oplus_{T^*G} , the identity section $\tilde{\epsilon}$ and the inversion \tilde{i} are defined as follows,

$$\begin{aligned} \tilde{\beta}(\mu_g)(X) &= \mu_g((T_{\epsilon(\beta(g))}l_g)(X)), \text{ for } \mu_g \in T_g^*G \text{ and } X \in A_{\beta(g)}G, \\ \tilde{\alpha}(\nu_h)(Y) &= \nu_h((T_{\epsilon(\alpha(h))}r_h)(Y - (T_{\epsilon(\alpha(h))}(\epsilon \circ \beta))(Y))), \\ &\quad \text{for } \nu_h \in T_h^*G \text{ and } Y \in A_{\alpha(h)}G, \\ (\mu_g \oplus_{T^*G} \nu_h)(T_{(g,h)}m(X_g, Y_h)) &= \mu_g(X_g) + \nu_h(Y_h), \\ &\quad \text{for } (X_g, Y_h) \in T_{(g,h)}G_2, \\ \tilde{\epsilon}(\mu_x)(X_{\epsilon(x)}) &= \mu_x(X_{\epsilon(x)} - (T_{\epsilon(x)}(\epsilon \circ \alpha))(X_{\epsilon(x)})), \\ &\quad \text{for } \mu_x \in A_x^*G \text{ and } X_{\epsilon(x)} \in T_{\epsilon(x)}G, \\ \tilde{i}(\mu_g)(X_{g^{-1}}) &= -\mu_g((T_{g^{-1}}i)(X_{g^{-1}})), \text{ for } \mu_g \in T_g^*G \text{ and } X_{g^{-1}} \in T_{g^{-1}}G. \end{aligned} \quad (2.4)$$

Note that $\tilde{\epsilon}(A^*G)$ is just the conormal bundle of $M \cong \epsilon(M)$ as a submanifold of G . In addition, A^*G is endowed with a linear Poisson structure. For this Poisson structure on A^*G , $\tilde{\alpha} : T^*G \rightarrow A^*G$ is a Poisson map and $\tilde{\beta} : T^*G \rightarrow A^*G$ is an anti-Poisson (for more details, see [5, 13]).

3. DISCRETE DYNAMICS IN IMPLICIT FORM

Motivated by the studies of implicit differential equations in [14, 15, 16] and its generalization to Lie algebroids in [11], we introduce the notion of implicit difference equations on Lie groupoids and the corresponding notion of integrability.

Definition 3.1. *An **implicit difference equation** on a Lie groupoid $G \rightrightarrows M$ is a submanifold E of G .*

An **admissible sequence** on the Lie groupoid $G \rightrightarrows M$ is a mapping $\gamma_G : I \cap \mathbb{Z} \rightarrow G$ such that $\beta(\gamma_G(i)) = \alpha(\gamma_G(i+1))$ for all $i, i+1 \in I \cap \mathbb{Z}$. Here I is an interval on \mathbb{R} .

A **solution** of an implicit difference equation $E \subset G$ is an admissible sequence $\gamma_G : I \cap \mathbb{Z} \rightarrow G$ on G such that $\gamma_G(i) \in E$, for all $i \in I \cap \mathbb{Z}$.

Let E be a submanifold of the Lie groupoid $G \rightrightarrows M$. Then E is said to be

- 1) **forward integrable at** $g \in E$ if there is a solution $\gamma_G : \mathbb{Z}^+ \rightarrow E \subseteq G$ with $\gamma_G(0) = g$. Here, $\mathbb{Z}^+ = \{n \in \mathbb{Z} \mid n \geq 0\}$.
- 2) **backward integrable at** $g \in E$ if there is a solution $\gamma_G : \mathbb{Z}^- \rightarrow E \subseteq G$ with $\gamma_G(0) = g$. Here, $\mathbb{Z}^- = \{n \in \mathbb{Z} \mid n \leq 0\}$.
- 3) **integrable at** $g \in E$ if if there is a solution $\gamma_G : \mathbb{Z} \rightarrow E \subseteq G$ with $\gamma_G(0) = g$, that is, it is backward and forward integrable at $g \in E$.

If these conditions hold for all g , we say that E is forward integrable, backward integrable or integrable, respectively.

Proposition 3.2. *Let E be a submanifold of the Lie groupoid $G \rightrightarrows M$. Then,*

- 1) *E is forward integrable if and only if for each $g \in E$ exists at least an $h \in E$ such that $(g, h) \in G_2$ or, equivalently, $E \subseteq \beta^{-1}(\alpha(E))$.*
- 2) *E backward integrable if and only if for each $g \in E$ exists at least an $h' \in E$ such that $(h', g) \in G_2$ or, equivalently, $E \subseteq \alpha^{-1}(\beta(E))$.*
- 3) *E is integrable if and only if $E \subseteq \alpha^{-1}(\beta(E)) \cap \beta^{-1}(\alpha(E))$.*

Proof. 1) Suppose that E is forward integrable. Then, for all $g \in E$ there is an admissible sequence $\gamma_G : \mathbb{Z}^+ \rightarrow E$ such that $\gamma_G(0) = g$. So, we have that $h = \gamma_G(1) \in E$ satisfies $(g, h) \in G_2$.

Conversely, if $g \in E$ there exists a $g_1 \in E$ such that $(g, g_1) \in G_2$. Since $g_1 \in E$ and using again the hypothesis, there is $g_2 \in E$ satisfying $(g_1, g_2) \in G_2$. Therefore, we can construct a sequence $\{g_i\}_{i \in \mathbb{Z}^+} \subseteq E$ where $g_0 = g$ and $(g_i, g_{i+1}) \in G_2$. As a consequence, the sequence $\gamma_G : \mathbb{Z}^+ \rightarrow E$, $i \mapsto \gamma_G(i) = g_i$, is a solution of E with $\gamma_G(0) = g$.

With a similar argument we deduce 2), and 3) is a direct consequence of 1) and 2). ■

Example 3.3. Let M be a manifold and $\varphi \in C^\infty(M)$ be a diffeomorphism on M . Consider the implicit difference equation E_φ of the pair groupoid $M \times M$ given by

$$E_\varphi = \{(x, \varphi(x)) \in M \times M \mid x \in M\}.$$

Then, if $(x_0, \varphi(x_0)) \in E_\varphi$, the map $\gamma : \mathbb{Z} \rightarrow E_\varphi$ defined by

$$\gamma(i) = (\varphi^i(x_0), \varphi^{i+1}(x_0)), \quad \text{for } i \in \mathbb{Z},$$

where $\varphi^{-i} = (\varphi^{-1})^i$, is a solution of the implicit difference equation E_φ . Therefore, E_φ is integrable.

This example can be generalized to bisections on Lie groupoids. We recall that a bisection on a Lie groupoid is a map $\sigma : M \rightarrow G$ which is a section of α and such that $\beta \circ \sigma$ is a diffeomorphism. Given a bisection, σ , if $E_\sigma = \sigma(M)$ is the associated implicit difference equation on G then it is integrable, since for any $g = \sigma(x) \in E_\sigma$, the sequence

$$\gamma(i) = \sigma((\beta \circ \sigma)^i(x)), \quad \text{for } i \in \mathbb{Z},$$

is a solution with $\gamma(0) = g$.

The sets

$$\begin{aligned} \tilde{E}_f &= \{g \in E \mid E \text{ is forward integrable at } g\}, \\ \tilde{E}_b &= \{g \in E \mid E \text{ is backward integrable at } g\}, \\ \tilde{E}_{fb} &= \{g \in E \mid E \text{ is integrable at } g\}, \end{aligned}$$

are called the **forward integrable**, **backward integrable** and **integrable parts** of E , respectively. The implicit difference equation E is integrable (resp. forward integrable or backward integrable) if $E = \tilde{E}_{fb}$ (resp. $E = \tilde{E}_f$ or $E = \tilde{E}_b$). Note that if $E \subseteq E'$ then $\tilde{E}_f \subset \tilde{E}'_f$, $\tilde{E}_b \subset \tilde{E}'_b$ and $\tilde{E}_{fb} \subset \tilde{E}'_{fb}$.

Proposition 3.4. *Let E be a submanifold of a Lie groupoid $G \rightrightarrows M$. Assume that \tilde{E}_{fb} , the integrable part of E , is a submanifold. Then, \tilde{E}_{fb} is integrable.*

Proof. If $g \in \tilde{E}_{fb}$ then there exists an admissible sequence $\gamma_G : \mathbb{Z} \rightarrow E \subseteq G$ with $\gamma_G(0) = g$. We have that $\gamma_G(\mathbb{Z}) \subseteq \tilde{E}_{fb}$. Indeed, for all $\tilde{n} \in \mathbb{Z}$ there is an admissible sequence $\gamma_{G, \tilde{n}} : \mathbb{Z} \rightarrow E$ given by

$$\gamma_{G, \tilde{n}}(i) = \gamma_G(i + \tilde{n}), \quad \text{for all } i \in \mathbb{Z}.$$

■

A similar result holds for \tilde{E}_f and \tilde{E}_b . Thus, \tilde{E}_f and \tilde{E}_b are forward and backward integrable, respectively.

Now, it is easy to prove the following

Proposition 3.5. *Let E be a submanifold of a Lie groupoid $G \rightrightarrows M$.*

- (i) *If \tilde{E}_f is the forward integrable part of E and E' is a submanifold of G such that $\tilde{E}_f \subseteq E' \subseteq E$, then \tilde{E}_f is the forward integrable part of E' .*
- (ii) *If \tilde{E}_b is the backward integrable part of E and E' is a submanifold of G such that $\tilde{E}_b \subseteq E' \subseteq E$, then \tilde{E}_b is the backward integrable part of E' .*
- (iii) *If \tilde{E}_{fb} is the integrable part of E and E' is a submanifold of G such that $\tilde{E}_{fb} \subseteq E' \subseteq E$, then \tilde{E}_{fb} is the integrable part of E' .*

From Proposition 3.2, we have the following obvious relations

$$\begin{aligned}\tilde{E}_f &\subseteq \beta^{-1}(\alpha(\tilde{E}_f)) \subseteq \beta^{-1}(\alpha(E)) \\ \tilde{E}_b &\subseteq \alpha^{-1}(\beta(\tilde{E}_b)) \subseteq \alpha^{-1}(\beta(E)) \\ \tilde{E}_{fb} &\subseteq \beta^{-1}(\alpha(\tilde{E}_{fb})) \cap \alpha^{-1}(\beta(\tilde{E}_{fb})) \subseteq \beta^{-1}(\alpha(E)) \cap \alpha^{-1}(\beta(E))\end{aligned}$$

Therefore,

$$\begin{aligned}\tilde{E}_f &\subseteq E \cap \beta^{-1}(\alpha(E)) \\ \tilde{E}_b &\subseteq E \cap \alpha^{-1}(\beta(E)) \\ \tilde{E}_{fb} &\subseteq E \cap \beta^{-1}(\alpha(E)) \cap \alpha^{-1}(\beta(E))\end{aligned}$$

Propositions 3.2 and 3.5 suggest a method for extracting the integrable part of an implicit difference equation under some regularity hypothesis.

Let $E \subset G$ be an implicit difference equation of a Lie groupoid $G \rightrightarrows M$.

For forward integrable part. Consider the sequence of sets of M

$$C_f^0 = \alpha(E), \quad C_f^1 = \alpha(E \cap \beta^{-1}(C_f^0)), \dots, C_f^k = \alpha(E \cap \beta^{-1}(C_f^{k-1})), \dots$$

and the sequence subsets of G

$$E_f^0 = E, \quad E_f^1 = E \cap \beta^{-1}(C_f^0), \dots, E_f^k = E \cap \beta^{-1}(C_f^{k-1}), \dots$$

Observe that $\alpha(E_f^k) = C_f^k$, $C_f^k \subseteq C_f^{k-1}$ and $E_f^k \subseteq E_f^{k-1}$.

For backward integrable part. Consider the sequence of sets of M

$$D_b^0 = \beta(E), \quad D_b^1 = \beta(E \cap \alpha^{-1}(D_b^0)), \dots, D_b^k = \beta(E \cap \alpha^{-1}(D_b^{k-1})), \dots$$

and the sequence of subsets of G

$$E_b^0 = E, \quad E_b^1 = E \cap \alpha^{-1}(D_b^0), \dots, E_b^k = E \cap \alpha^{-1}(D_b^{k-1}), \dots$$

Also then $D_b^k \subseteq D_b^{k-1}$ and $E_b^k \subseteq E_b^{k-1}$.

For integrable part. Consider the sequence of sets of M

$$\begin{aligned}C_{fb}^0 &= \alpha(E), \quad D_{fb}^0 = \beta(E), \\ C_{fb}^1 &= \alpha(E \cap \alpha^{-1}(D_{fb}^0) \cap \beta^{-1}(C_{fb}^0)), \quad D_{fb}^1 = \beta(E \cap \alpha^{-1}(D_{fb}^0) \cap \beta^{-1}(C_{fb}^0)) \\ C_{fb}^k &= \alpha(E \cap \alpha^{-1}(D_{fb}^{k-1}) \cap \beta^{-1}(C_{fb}^{k-1})), \quad D_{fb}^k = \beta(E \cap \alpha^{-1}(D_{fb}^{k-1}) \cap \beta^{-1}(C_{fb}^{k-1})) \dots\end{aligned}$$

and the sequence of subsets of G

$$\begin{aligned}E_{fb}^0 &= E, E_{fb}^1 = E \cap \alpha^{-1}(D_{fb}^0) \cap \beta^{-1}(C_{fb}^0), \dots \\ E_{fb}^k &= E \cap \alpha^{-1}(D_{fb}^{k-1}) \cap \beta^{-1}(C_{fb}^{k-1}), \dots\end{aligned}$$

Then $C_{fb}^k \subseteq C_{fb}^{k-1}$, $D_{fb}^k \subseteq D_{fb}^{k-1}$ and $E_{fb}^k \subseteq E_{fb}^{k-1}$.

It may happen that after a finite number of steps, the sets in two consecutive steps in the different sequences are equal:

$$\begin{aligned}C_f^{k_f} &= C_f^{k_f+1} = \bar{C}_f, \quad D_b^{k_b} = D_b^{k_b+1} = \bar{D}_b, \\ C_{fb}^{k_{fb}} &= C_{fb}^{k_{fb}+1} = \bar{C}_{fb}, \quad D_{fb}^{k_{fb}} = D_{fb}^{k_{fb}+1} = \bar{D}_{fb}.\end{aligned}$$

Then

$$\begin{aligned} E_f^{k_f+1} &= E_f^{k_f+2} = \bar{E}_f, & E_b^{k_b+1} &= E_b^{k_b+2} = \bar{E}_b, \\ E_{fb}^{k_{fb}+1} &= E_{fb}^{k_{fb}+2} = \bar{E}_{fb}. \end{aligned}$$

It is straightforward to see that if this happens then the subsequent steps in the different sequences are all equal.

Moreover, observe that $\alpha(\bar{E}_f) = \bar{C}_f$, $\beta(\bar{E}_b) = \bar{D}_b$, $\alpha(\bar{E}_{fb}) = \bar{C}_{fb}$ and $\beta(\bar{E}_{fb}) = \bar{D}_{fb}$.

Theorem 3.6. *The forward integrable part of E is \bar{E}_f , the backward integrable part is \bar{E}_b and the integrable part of E is \bar{E}_{fb} .*

Proof. Let $g \in \bar{E}_f = E_f^{k_f} = E_f^{k_f+1}$. Then, since $E_f^{k_f+1} = E \cap \beta^{-1}(C_f^{k_f})$, we have that $\beta(g) \in C_f^{k_f} = \alpha(\bar{E}_f)$, that is, there exists $h \in E_f^{k_f+1} = E_f^{k_f} = \bar{E}_f$ such that $\beta(g) = \alpha(h)$. Therefore, using Proposition 3.2, we have that E is forward integrable at g .

Conversely, suppose that E is forward integrable at g . Then, there exists an admissible sequence $\gamma_G : \mathbb{Z}^+ \rightarrow E \subseteq G$ such that $\gamma_G(0) = g$. Denote $\gamma_G(i)$ by g_i , for all $i \in \mathbb{Z}^+$. Then, $\beta(g_{k_f}) = \alpha(g_{k_f+1})$ and $g_{k_f} \in E \cap \beta^{-1}(C_f^0) = E_f^1$, which implies that $\alpha(g_{k_f}) \in C_f^1$. Now, since $\beta(g_{k_f-1}) = \alpha(g_{k_f})$, we deduce that $g_{k_f-1} \in E_f^2$. If we repeat this argument k_f times, we conclude that $g = g_0 \in E_f^{k_f+1} = E_f^{k_f+2} = \bar{E}_f$.

The corresponding results for backward integrability and integrability are proved in a similar way. ■

Remark 3.7. We have to stress that the concept of implicit difference equation can be generalized to the setting of a pair of manifolds G and M endowed with two surjective submersions $\alpha, \beta : G \rightarrow M$. The corresponding notions of admissible sequences, solutions, integrable equations and the integrability algorithm can be straightforwardly generalized. This more general approach will be useful in Section 4.4. ◇

Remark 3.8. For some concrete applications (see, for instance, the next example) instead of a unique subset E of the Lie groupoid $G \rightrightarrows M$, we have a sequence of subsets $\{E_k\}_{0 \leq k \leq \infty}$. A generalization of this type is also useful for discretization of time-dependent systems. It is not hard to extend our setting for this situation and also to extract the integrable part of a sequence of this type. For example, the forward integrable part is extracted using the following procedure. Consider the k -sequence of sets of M defined for $k \geq 0$:

$$[C_k]_f^0 = \alpha(E_k), \quad [C_k]_f^1 = \alpha(E_k \cap \beta^{-1}([C_{k+1}]_f^0)), \dots, [C_k]_f^i = \alpha(E_k \cap \beta^{-1}([C_{k+1}]_f^{i-1})),$$

and the subsets

$$[E_k]_f^0 = E_k, \quad [E_k]_f^1 = E_k \cap \beta^{-1}([C_{k+1}]_f^0), \dots, [E_k]_f^i = E_k \cap \beta^{-1}([C_k]_f^{i-1}).$$

If the algorithm stabilizes in a sequence of subsets $\{\overline{[E_k]_f}\}$ we will call this set the backward integrable part of the sequence $\{E_k\}$. ◇

Example 3.9. Finite difference methods for time-dependent linear differential-algebraic equations The proposed algorithm is also useful for numerical methods for differential-algebraic equations (DAEs). These systems of equations appear in a great number of applications: mechanical systems with constraints, electrical networks, chemical reacting systems... See, for instance, [3, 22]. These types of systems take the form $F(t, x, \dot{x}) = 0$ including as a particular example the explicit differential equations. The system that we will study are the so-called time-dependent linear differential-algebraic equations:

$$A(t)\dot{x} + B(t)x = b(t), \quad A(t), B(t) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n), \quad b(t) \in \mathbb{R}^n, \quad t \in \mathbb{R}$$

Consider an explicit Euler approximation to this system

$$A_k \frac{x_{k+1} - x_k}{h} + B_k x_k = b_k$$

where $h > 0$ is the step size and $A_k = A(t_0 + kh)$, $B_k = B(t_0 + kh)$ and $b_k = b(t_0 + kh)$ and x_0, t_0 initial conditions. Define the subsets of the pair groupoid $\mathbb{R}^n \times \mathbb{R}^n$

$$E_k = \{(x_k, x_{k+1}) \in \mathbb{R}^n \times \mathbb{R}^n \mid A_k \frac{x_{k+1} - x_k}{h} + B_k x_k = b_k\}$$

We consider V_k the image of the linear map associated with A_k . Then, $V_k = \mathbb{R}^n$ (in this case, $x_{k+1} = x_k + hA_k^{-1}(b_k - B_k x_k)$) or there exists a matrix $Q_k \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ such that the kernel of its linear map is just V_k , i.e., $Q_k A_k = \mathbf{0}$. Then, one may easily prove that

$$[C_k]_f^0 = \{x \in \mathbb{R}^n \mid Q_k B_k x - Q_k b_k = 0\}.$$

Therefore,

$$\begin{aligned} [E_k]_f^1 &= \{(x_k, x_{k+1}) \in \mathbb{R}^n \times \mathbb{R}^n \mid A_k \frac{x_{k+1} - x_k}{h} + B_k x_k = b_k, \\ &\quad Q_{k+1} B_{k+1} x_{k+1} - Q_{k+1} b_{k+1} = 0\} \\ &= \{(x_k, x_{k+1}) \in \mathbb{R}^n \times \mathbb{R}^n \mid Q_{k+1} B_{k+1} x_{k+1} - Q_{k+1} b_{k+1} = 0, \\ &\quad (A_k + Q_{k+1} B_{k+1}) x_{k+1} = (A_k - h B_k) x_k + h b_k + Q_{k+1} b_{k+1}, \} \end{aligned}$$

It is possible to continue the algorithm, considering $\alpha([E_k]_f^1) = [C_k]_f^1$, but in many cases of interest (as, for instance, discretizations of index one DAEs) we have that the matrix $A_k + Q_{k+1} B_{k+1}$ is regular, then

$$x_{k+1} = (A_k + Q_{k+1} B_{k+1})^{-1} ((A_k - h B_k) x_k + h b_k + Q_{k+1} b_{k+1}).$$

In this case, $\alpha([E_k]_f^1) = [C_k]_f^1 = [C_k]_f^0$ and the algorithm stops in this step (see [23] for more details).

4. IMPLICIT SYSTEMS DEFINED BY DISCRETE LAGRANGIAN FUNCTIONS

4.1. Implicit systems defined by discrete Lagrangian functions. Lagrangian submanifolds. Let $L : G \rightarrow \mathbb{R}$ be a discrete Lagrangian on a Lie groupoid $G \rightrightarrows M$. Then, $S_L = dL(G) \subseteq T^*G$ is clearly a Lagrangian submanifold of the symplectic manifold T^*G and S_L is also an implicit difference equation on the cotangent groupoid $T^*G \rightrightarrows A^*G$. If we apply the forward integrability algorithm to S_L , then there is an interesting relation with the discrete Euler-Lagrange equations for L (see

[18, 25] and the Appendix A for more details on discrete Lagrangian Mechanics on Lie groupoids), which we describe in this section.

Proposition 4.1. *Let $G \rightrightarrows M$ be a Lie groupoid and $L : G \rightarrow \mathbb{R}$ be a discrete Lagrangian on G . The first step, $(S_L)_f^1$, to obtain the forward integrable part of $S_L = dL(G)$ are the points $dL(g) \in S_L$ such that there exists an element $h \in G$, satisfying $(g, h) \in G_2$ and, in addition, (g, h) is a solution of the discrete Euler-Lagrange equations, that is,*

$$d[L \circ l_g + L \circ r_h \circ i](\epsilon(x))|_{A_x G} = 0,$$

where $x = \beta(g) = \alpha(h)$.

As a consequence, if $\tilde{\beta}$ (resp. $\tilde{\alpha}$) is the target (resp. source) map of the cotangent groupoid $T^*G \rightrightarrows A^*G$ defined in (2.4), then

$$\begin{aligned} (S_L)_f^1 &= \{(g, h) \in G_2 \mid (g, h) \text{ is a solution of the discrete Euler-Lagrange eqns.}\} \\ &= \{(g, h) \in G_2 \mid \tilde{\beta}(dL(g)) = \tilde{\alpha}(dL(h))\} \end{aligned}$$

Proof. Using the definition of the source map $\tilde{\alpha} : T^*G \rightarrow A^*G$ (see (2.4)) and the fact that

$$X - T_{\epsilon(x)}(\epsilon \circ \beta)(X) = -T_{\epsilon(x)}i(X), \text{ for } X \in A_x G,$$

we have that

$$C_f^0 = \tilde{\alpha}(S_L) = \{a^* \in A^*G \mid a^* = -d(L \circ r_h \circ i)(\epsilon(\alpha(h))) \in A_{\alpha(h)}^*G, \text{ for some } h \in G\}.$$

Therefore, $(S_L)_f^1 = S_L \cap (\tilde{\beta})^{-1}(C_f^0)$ is the set of points $dL(g) \in S_L$ such that there exists $h \in G$, with $(g, h) \in G_2$ and

$$d[L \circ l_g + L \circ r_h \circ i](\epsilon(x))|_{A_x G} = 0.$$

■

Let $L : G \rightarrow \mathbb{R}$ be a hyperregular discrete Lagrangian function (see Appendix A). This implies that the discrete Lagrangian evolution operator $\Upsilon_L : G \rightarrow G$ is given by $\Upsilon_L = (\mathbb{F}^- L)^{-1} \circ (\mathbb{F}^+ L)$. Then, if $g \in G$, the map $\Upsilon_{T^*G} : \mathbb{Z} \rightarrow S_L = dL(G) \subseteq T^*G$ defined by

$$\Upsilon_{T^*G}(i) = dL(\Upsilon_L^i(g)), \quad \text{for } i \in \mathbb{Z},$$

is a solution of the implicit difference equation S_L and $\Upsilon_{T^*G}(0) = dL(g)$.

Thus, we have proved the following result.

Corollary 4.2. *Let $G \rightrightarrows M$ be a Lie groupoid and $L : G \rightarrow \mathbb{R}$ be a discrete hyperregular Lagrangian function on G . Then, the implicit difference equation $S_L = dL(G)$ is integrable.*

Under the same hypotheses as in Corollary 4.2, we may define the map

$$\begin{aligned} \Phi : G &\xrightarrow{(Id, \Upsilon_L)} G \times G \xrightarrow{(\mathbb{F}^- L, \mathbb{F}^- L)} A^*G \times A^*G \\ g &\longmapsto (g, \Upsilon_L(g)) \longmapsto (\mathbb{F}^- L(g), \mathbb{F}^+ L(g)) \end{aligned}$$

A direct computation, using (A.3), shows that $\Phi(G) = (\tilde{\alpha}, \tilde{\beta})(S_L)$.

On the other hand, if $\tilde{\Upsilon}_L = \mathbb{F}^+L \circ (\mathbb{F}^-L)^{-1} : A^*G \rightarrow A^*G$ is the discrete Hamiltonian evolution operator then one observes that

$$\text{graph } \tilde{\Upsilon}_L = \Phi(G) = (\tilde{\alpha}, \tilde{\beta})(S_L).$$

Using that S_L is a Lagrangian submanifold of T^*G and that $(\tilde{\alpha}, \tilde{\beta}) : T^*G \rightarrow A^*G \times \overline{A^*G}$ is a Poisson map, $\overline{A^*G}$ denoting A^*G endowed with the linear Poisson structure changed of sign, then $(\tilde{\alpha}, \tilde{\beta})(S_L)$ is a coisotropic submanifold of $A^*G \times \overline{A^*G}$ (this is a particular case of [28, Corollary 2.2.5]).

From the previous discussion, we can conclude the following result (see also [29]).

Proposition 4.3. *Let $G \rightrightarrows M$ be a Lie groupoid and $L : G \rightarrow \mathbb{R}$ be a discrete hyperregular Lagrangian on G . Then, the discrete Hamiltonian evolution operation $\tilde{\Upsilon}_L$ preserves the Poisson structure on A^*G .*

4.2. Example: the pair groupoid. Let Q be a manifold and $L : Q \times Q \rightarrow \mathbb{R}$ be a discrete Lagrangian. We know that $dL(Q \times Q)$ is a Lagrangian submanifold of $T^*(Q \times Q)$. On the other hand, $T^*(Q \times Q) \rightrightarrows T^*Q$ is a symplectic groupoid with the canonical symplectic structure of $T^*(Q \times Q)$. The structural maps of $T^*(Q \times Q) \rightrightarrows T^*Q$ are

$$\begin{aligned} \tilde{\beta} : T^*(Q \times Q) &\rightarrow T^*Q, & (\gamma_{q_0}, \gamma_{q_1}) &\mapsto \gamma_{q_1}, \\ \tilde{\alpha} : T^*(Q \times Q) &\rightarrow T^*Q, & (\gamma_{q_0}, \gamma_{q_1}) &\mapsto -\gamma_{q_0}, \\ \tilde{m} : T^*(Q \times Q)_2 &\rightarrow T^*Q, & ((\gamma_{q_0}, \gamma_{q_1}), (-\gamma_{q_1}, \gamma_{q_2})) &\mapsto (\gamma_{q_0}, \gamma_{q_2}), \\ \tilde{i} : T^*(Q \times Q) &\rightarrow T^*(Q \times Q), & (\gamma_{q_0}, \gamma_{q_1}) &\mapsto (-\gamma_{q_1}, -\gamma_{q_0}), \\ \tilde{\epsilon} : T^*Q &\rightarrow T^*(Q \times Q), & \gamma_q &\mapsto (\gamma_q, -\gamma_q). \end{aligned}$$

Moreover,

$$\begin{aligned} \tilde{\beta}(dL(q_0, q_1)) &= D_2L(q_0, q_1), \\ \tilde{\alpha}(dL(q_1, q_2)) &= -D_1L(q_1, q_2), \end{aligned}$$

for $(q_0, q_1), (q_1, q_2) \in Q \times Q$. Thus, the discrete Euler-Lagrange equations for the points $(q_0, q_1), (q_1, q_2)$, that is $D_2L(q_0, q_1) + D_1L(q_1, q_2) = 0$, are equivalent to

$$\tilde{\beta}(dL(q_0, q_1)) = \tilde{\alpha}(dL(q_1, q_2)).$$

Therefore, we obtain the following result (which can also be seen as a consequence of Proposition 4.1)

Corollary 4.4. *Let Q be a smooth manifold and $L : Q \times Q \rightarrow \mathbb{R}$ be a discrete Lagrangian function. Then, there exists a bijective correspondence between composable points in $dL(Q \times Q)$ and solutions of the discrete Euler-Lagrange equations.*

Remark 4.5. $T^*(Q \times Q) \rightrightarrows T^*Q$ is a Lie groupoid which is symplectomorphic to the pair groupoid $T^*Q \times T^*Q \rightrightarrows T^*Q$ when we consider the symplectic form $(-\Omega_Q, \Omega_Q)$ on $T^*Q \times T^*Q$.

The isomorphism is given by

$$\begin{aligned} \Psi : T^*(Q \times Q) &\rightarrow T^*Q \times T^*Q \\ (\gamma_{q_0}, \gamma_{q_1}) &\mapsto (-\gamma_{q_0}, \gamma_{q_1}) \end{aligned} \tag{4.1}$$

Note that $(\Psi \circ dL)(q_0, q_1) = (-D_1L(q_0, q_1), D_2L(q_0, q_1))$. \diamond

Now, suppose that $L : Q \times Q \rightarrow \mathbb{R}$ is hyperregular. Then

$$\Upsilon_L = (\mathbb{F}^- L)^{-1} \circ \mathbb{F}^+ L : Q \times Q \rightarrow Q \times Q \quad (4.2)$$

is a discrete Lagrangian evolution operator. In addition, we have that the image of the map

$$\begin{aligned} Q \times Q &\xrightarrow{(Id, \Upsilon_L)} (Q \times Q) \times (Q \times Q) \xrightarrow{\mathbb{F}^- L \times \mathbb{F}^- L} T^*Q \times T^*Q \\ (q_0, q_1) &\mapsto ((q_0, q_1), \Upsilon_L(q_0, q_1)) \mapsto (-D_1 L(q_0, q_1), D_2 L(q_0, q_1)) \end{aligned}$$

is just the Lagrangian submanifold $\Psi \circ dL(Q \times Q)$.

On the other hand, we can consider the discrete Hamiltonian evolution operator $\tilde{\Upsilon}_L = \mathbb{F}^+ L \circ (\mathbb{F}^- L)^{-1} : T^*Q \rightarrow T^*Q$. A direct computation proves that

$$\Phi(Q \times Q) = (\Psi \circ dL)(Q \times Q) = (Id, \tilde{\Upsilon}_L)(T^*Q).$$

We remark that, since $\tilde{\Upsilon}_L : T^*Q \rightarrow T^*Q$ is a symplectomorphism, we deduce $(Id, \tilde{\Upsilon}_L)(T^*Q)$ is a Lagrangian submanifold of $(T^*Q \times T^*Q, (-\Omega_Q, \Omega_Q))$.

4.3. Example: a discrete singular Lagrangian. Consider the discretization of the degenerate Lagrangian $L : T\mathbb{R}^2 \rightarrow \mathbb{R}$ given by $L(x, y, \dot{x}, \dot{y}) = \frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2\dot{y}$ (see [2]). Take, for instance, a consistent discretization of this lagrangian:

$$L_d^h : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}, \quad L_d^h(x_1, y_1, x_2, y_2) = \frac{1}{2} \left(\frac{x_2 - x_1}{h} \right)^2 + \frac{1}{2} x_1^2 y_1$$

Then $S_{L_d^h} \subset T^*(\mathbb{R}^2 \times \mathbb{R}^2)$ defined by

$$S_{L_d^h} = \left\{ (x_1, y_1, x_2, y_2; -\frac{x_2 - x_1}{h^2} + x_1 y_1, \frac{1}{2} x_1^2, \frac{x_2 - x_1}{h^2}, 0) \mid (x_1, y_1, x_2, y_2) \in \mathbb{R}^4 \right\}$$

The algorithm produces the sequence:

$$\begin{aligned} C_f^1 &= \tilde{\alpha}(S_{L_d^h}) = \{(x, y; \frac{\tilde{x} - x}{h^2} - xy, -\frac{1}{2}x^2) \mid (x, y, \tilde{x}) \in \mathbb{R}^3\} \subset T^*\mathbb{R}^2 \\ (S_{L_d^h}^1)_f &= \left\{ (x_1, y_1, 0, y_2; \frac{x_1}{h^2} + x_1 y_1, \frac{1}{2} x_1^2, -\frac{x_1}{h^2}, 0) \mid (x_1, y_1, y_2) \in \mathbb{R}^3 \right\} \\ C_f^2 &= \tilde{\alpha}((S_{L_d^h}^1)_f) = \{(x, y; -\frac{x}{h^2} - xy, -\frac{1}{2}x^2) \mid (x, y) \in \mathbb{R}^2\} \subset T^*\mathbb{R}^2 \\ (S_{L_d^h}^2)_f &= \{(0, y_1, 0, y_2; 0, 0, 0, 0) \mid (y_1, y_2) \in \mathbb{R}^2\} \\ C_f^3 &= \tilde{\alpha}((S_{L_d^h}^2)_f) = \{(0, y; 0, 0) \mid y \in \mathbb{R}\} \subset T^*\mathbb{R}^2 \end{aligned}$$

Since $(S_{L_d^h}^3)_f = (S_{L_d^h}^2)_f$ then the algorithm terminates with $(\overline{S_{L_d^h}})_f = (S_{L_d^h}^2)_f$, which is the forward integrable part of the implicit difference equation $S_{L_d^h}$.

4.4. Implicit systems defined by discrete nonholonomic Lagrangian systems. Let $(L, \mathcal{M}_c, \mathcal{D}_c)$ be a discrete nonholonomic Lagrangian system. Consider the submanifold $S_{(L, \mathcal{M}_c)} \subseteq T^*G$ given by $S_{(L, \mathcal{M}_c)} = dL(\mathcal{M}_c)$, that is,

$$S_{(L, \mathcal{M}_c)} = \{ dL(i_{\mathcal{M}_c}(g)) \mid g \in \mathcal{M}_c \}.$$

Composing the source and the target maps $\tilde{\alpha}$ and $\tilde{\beta}$ of the cotangent groupoid $T^*G \rightrightarrows A^*G$ as defined in Equation (2.4) with $i_{\mathcal{D}_c}^* : AG^* \rightarrow \mathcal{D}_c^*$, the dual map of the canonical inclusion $i_{\mathcal{D}_c} : \mathcal{D}_c \rightarrow AG$, we have the maps

$$\begin{aligned} \tilde{\alpha}_{\mathcal{D}_c} &= i_{\mathcal{D}_c}^* \circ \tilde{\alpha} : T^*G \rightarrow \mathcal{D}_c^* \\ \tilde{\beta}_{\mathcal{D}_c} &= i_{\mathcal{D}_c}^* \circ \tilde{\beta} : T^*G \rightarrow \mathcal{D}_c^*. \end{aligned} \quad (4.3)$$

If we apply the forward integrability algorithm to the implicit difference equation $S_{(L, \mathcal{M}_c)}$ in T^*G , but changing the maps $\tilde{\alpha}$ and $\tilde{\beta}$ for the new maps $\tilde{\alpha}_{\mathcal{D}_c}$ and $\tilde{\beta}_{\mathcal{D}_c}$ (see Remark 3.7), then the first step of the algorithm yields a close relation with the discrete non-holonomic Euler-Lagrange equations. Indeed, if we have the following result.

Proposition 4.6. *Let $G \rightrightarrows M$ be a Lie groupoid and $(L, \mathcal{M}_c, \mathcal{D}_c)$ be a discrete nonholonomic Lagrangian system on G . The first step, $(S_{(L, \mathcal{M}_c)})_f^1$, to obtain the forward integrable part of $S_{(L, \mathcal{M}_c)} = dL(\mathcal{M}_c)$ for the maps $\tilde{\alpha}_{\mathcal{D}_c} : T^*G \rightarrow \mathcal{D}_c^*$ and $\tilde{\beta}_{\mathcal{D}_c} : T^*G \rightarrow \mathcal{D}_c^*$, defined in (4.3), are the points $dL(g) \in S_{(L, \mathcal{M}_c)}$ such that there exists an element $h \in \mathcal{M}_c$, satisfying $(g, h) \in G_2$ and, in addition, (g, h) is a solution of the discrete non-holonomic Euler-Lagrange equations, that is,*

$$d[L \circ l_g + L \circ r_h \circ i](\epsilon(x))|_{(\mathcal{D}_c)_x} = 0,$$

where $x = \beta(g) = \alpha(h)$.

In other words,

$$\begin{aligned} (S_L)_f^1 &= \{(g, h) \in G_2 \cap (\mathcal{M}_c \times \mathcal{M}_c) \mid \\ &\quad (g, h) \text{ is a solution of the discrete nonholonomic Euler-Lagrange eqns.}\} \\ &= \{(g, h) \in G_2 \cap (\mathcal{M}_c \times \mathcal{M}_c) \mid \tilde{\beta}_{\mathcal{D}_c}(dL(g)) = \tilde{\alpha}_{\mathcal{D}_c}(dL(h))\} \end{aligned}$$

Proof. The proof is analogous to that of Proposition 4.1, taking into account the version of the discrete nonholonomic Euler-Lagrange equations given by Equation (B.4). ■

4.5. Example: The discrete Chaplygin sleigh [7, 8]. To illustrate the results contained in the previous section, we will describe a discretization of the Chaplygin sleigh system (previously considered in [10]).

The Chaplygin sleigh system describes the motion of a rigid body sliding on a horizontal plane. The body is supported at three points, two of which slide freely without friction while the third is a knife edge, a constraint that allows no motion orthogonal to this edge (see [21]).

The configuration space of this system is the group $SE(2)$ of Euclidean motions of \mathbb{R}^2 . An element $A \in SE(2)$ is represented by a matrix

$$A = \begin{pmatrix} \cos \theta & -\sin \theta & x \\ \sin \theta & \cos \theta & y \\ 0 & 0 & 1 \end{pmatrix} \quad \text{with } \theta, x, y \in \mathbb{R}.$$

Thus, (θ, x, y) are local coordinates on $SE(2)$.

A basis of the Lie algebra $\mathfrak{se}(2) \cong \mathbb{R}^3$ of $SE(2)$ is given by

$$e = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

and we have that

$$[e, e_1] = e_2, \quad [e, e_2] = -e_1, \quad [e_1, e_2] = 0.$$

An element $\xi \in \mathfrak{se}(2)$ is of the form

$$\xi = \omega e + v_1 e_1 + v_2 e_2$$

The exponential map $\exp : \mathfrak{se}(2) \cong \mathbb{R}^3 \rightarrow SE(2)$ of $SE(2)$ is given by

$$\exp(\omega, v_1, v_2) = (\omega, v_1 \frac{\sin \omega}{\omega} + v_2 (\frac{\cos \omega - 1}{\omega}), -v_1 (\frac{\cos \omega - 1}{\omega}) + v_2 \frac{\sin \omega}{\omega}), \text{ if } \omega \neq 0,$$

and $\exp(0, v_1, v_2) = (0, v_1, v_2)$. The restriction of the exponential map to the open subset $U =]-\pi, \pi[\times \mathbb{R}^2 \subseteq \mathbb{R}^3 \cong \mathfrak{se}(2)$ is a diffeomorphism onto the open subset $\exp(U)$ of $SE(2)$.

A discretization of the Chaplygin sleigh may be constructed as follows (see [10] for more details):

First of all, the discrete Lagrangian $L : SE(2) \rightarrow \mathbb{R}$ is given by

$$L(A) = \frac{1}{2} \text{Tr} (A \mathbb{J} A^T) - \text{Tr} (A \mathbb{J}),$$

where \mathbb{J} is the matrix:

$$\mathbb{J} = \begin{pmatrix} (J/2) + ma^2 & mab & ma \\ mab & (J/2) + mb^2 & mb \\ ma & mb & m \end{pmatrix}$$

(see [8]). In terms of the coordinates (θ, x, y) , the discrete Lagrangian can be written as

$$\begin{aligned} L(\theta, x, y) &= (max + mby - ma^2 - mb^2 - J) \cos \theta + m(ay - bx) \sin \theta \\ &\quad + \frac{m}{2} ((x - a)^2 + (y - b)^2) + \frac{1}{2}(J - m) \end{aligned}$$

Second, the vector subspace \mathcal{D}_c of the Lie algebra $\mathfrak{se}(2)$ is given by

$$\mathcal{D}_c = \text{span} \{e, e_1\} = \{(\omega, v_1, v_2) \in \mathfrak{se}(2) \mid v_2 = 0\}.$$

Finally, the constraint submanifold \mathcal{M}_c of $SE(2)$ is $\mathcal{M}_c = \exp(U \cap \mathcal{D}_c)$, that is,

$$\begin{aligned} \mathcal{M}_c &= \{(\theta, x, y) \in SE(2) \mid -\pi < \theta < \pi, \theta \neq 0, (1 - \cos \theta)x - y \sin \theta = 0\} \\ &\quad \cup \{(0, x, 0) \in SE(2) \mid x \in \mathbb{R}\}. \end{aligned}$$

Let us calculate the set $S_{(L, \mathcal{M}_c)} = dL(\mathcal{M}_c)$. We have that $(\theta, x, y; p_\theta, p_x, p_y) \in T^*SE(2) \equiv \mathbb{R}^6$ belongs to $S_{(L, \mathcal{M}_c)}$ if it verifies the following equations:

$$\begin{aligned} p_\theta &= m(ay - bx) \cos \theta + (ma^2 + mb^2 + J - max - mby) \sin \theta, \\ p_x &= ma \cos \theta - mb \sin \theta + m(x - a), \\ p_y &= mb \cos \theta + ma \sin \theta + m(y - b), \end{aligned}$$

with $(\theta, x, y) \in \mathcal{M}_c$.

Moreover, using (2.4), we deduce that the maps $\tilde{\beta} : T^*SE(2) \rightarrow \mathfrak{se}(2)^*$ and $\tilde{\alpha} : T^*SE(2) \rightarrow \mathfrak{se}(2)^*$ are just the pullbacks by the left and right translations, respectively. Now, the mappings $\tilde{\alpha}_{\mathcal{D}_c} : T^*G \rightarrow \mathcal{D}_c^*$ and $\tilde{\beta}_{\mathcal{D}_c} : T^*G \rightarrow \mathcal{D}_c^*$ are:

$$\begin{aligned} \tilde{\alpha}_{\mathcal{D}_c}(p_\theta d\theta + p_x dx + p_y dy) &= (p_\theta - yp_x + xp_y)e^* + p_x e_1^* \\ \tilde{\beta}_{\mathcal{D}_c}(p_\theta d\theta + p_x dx + p_y dy) &= p_\theta e^* + (p_x \cos \theta + p_y \sin \theta)e_1^* \end{aligned}$$

where $p_\theta d\theta + p_x dx + p_y dy \in T_{(\theta, x, y)}^*G$ and $\{e^*, e_1^*\}$ is the dual basis of $\{e, e_1\}$ and therefore span $\{e^*, e_1^*\} = \mathcal{D}_c^*$.

Thus, the first step of the algorithm allows yields the set of points $(\theta_k, x_k, y_k) \in \mathcal{M}_c$, with $k \in \{1, 2\}$, such that

$$\tilde{\beta}_{\mathcal{D}_c}(dL(\theta_1, x_1, y_1)) = \tilde{\alpha}_{\mathcal{D}_c}(dL(\theta_2, x_2, y_2)),$$

which are just the discrete Euler-Poincaré-Suslov equations,

$$\begin{aligned} \begin{pmatrix} -ma \cos \theta_1 - mb \sin \theta_1 + ma \\ +mx_1 \cos \theta_1 + my_1 \sin \theta_1 \end{pmatrix} &= \begin{pmatrix} mx_2 + ma \cos \theta_2 \\ -mb \sin \theta_2 - ma \end{pmatrix} \\ \begin{pmatrix} m(ay_1 - bx_1) \cos \theta_1 - m(ax_1 + by_1) \sin \theta_1 \\ +(ma^2 + mb^2 + J) \sin \theta_1 \end{pmatrix} &= \begin{pmatrix} ma y_2 - mb x_2 \\ +(ma^2 + mb^2 + J) \sin \theta_2 \end{pmatrix} \end{aligned}$$

where $(\theta_k, x_k, y_k) \in \mathcal{M}_c$, with $k \in \{1, 2\}$.

4.6. An approach to implicit discrete Hamiltonian systems. Along the paper, we have focused our attention to the case of discrete lagrangian mechanics including this theory in the setting of discrete implicit systems. Of course, we can also adopt a dual point of view, that is, the hamiltonian formalism. In the continuous setting, the hamiltonian formalism is specified given a hamiltonian function H on A^*G equipped with its canonical linear Poisson bracket. The dynamics is given by the associated hamiltonian vector field X_H . In the sequel, we will show how to obtain a lagrangian submanifold of the symplectic groupoid $T^*G \rightrightarrows A^*G$ directly from the flow of X_H .

Assume that $G \rightrightarrows M$ is a Lie groupoid with corresponding algebroid $\tau : AG \rightarrow M$. Then, we have already explained (see (2.4)) that the cotangent bundle T^*G is a groupoid over A^*G . Moreover, there is a Poisson structure on A^*G (induced from the Lie algebroid structure on AG) such that the source map $\tilde{\alpha} : T^*G \rightarrow A^*G$ (resp. the target map $\tilde{\beta} : T^*G \rightarrow A^*G$) is a Poisson (resp. anti-Poisson) map.

Let $H : A^*G \rightarrow \mathbb{R}$ be a Hamiltonian function. Then, since A^*G is a Poisson manifold, we have the hamiltonian vector field X_H . On the other hand, consider the pull-back of the 1-form $dH \in \Omega^1(A^*G)$ by the source map $\tilde{\alpha}$. Using the canonical symplectic structure on T^*G , we have the corresponding Hamiltonian vector field

$X_{\tilde{\alpha}^*H}$. It is easy to see that both vector fields are $\tilde{\alpha}$ -related. Indeed, if Π_{A^*G} (resp. Π_G) denotes the Poisson structure on A^*G (resp. on T^*G), since $\tilde{\alpha}_*(\Pi_G) = \Pi_{A^*G}$,

$$X_H = \Pi_{A^*G}^\sharp(dH) = (\tilde{\alpha}_*\Pi_G)^\sharp(dH) = \tilde{\alpha}_*((\Pi_G)^\sharp(d\tilde{\alpha}^*H)) = \tilde{\alpha}_*(X_{\tilde{\alpha}^*H}),$$

where $\Pi^\sharp(dF)$ denotes contraction by dF , i.e., $\Pi^\sharp(dF)(\gamma) = \Pi(dF, \gamma)$ for any 1-form γ . Thus, we have that the following diagram is commutative

$$\begin{array}{ccc} T^*G & \xrightarrow{\varphi_H^t} & T^*G \\ \tilde{\alpha} \downarrow & & \downarrow \tilde{\alpha} \\ A^*G & \xrightarrow{\psi_H^t} & A^*G \end{array}$$

where φ_H^t (resp., ψ_H^t) is the flow of $X_{\tilde{\alpha}^*H}$ (resp., X_H). In addition, since $\varphi_H^t : T^*G \rightarrow T^*G$ is a symplectomorphism, $\tilde{i} : T^*G \rightarrow T^*G$ is an anti-symplectomorphism and $\tilde{\epsilon} : A^*G \rightarrow T^*G$ is a Lagrangian embedding, we deduce that the submanifold S_H^t defined by

$$S_H^t = \{\nu_g \in T^*G \mid \nu_g = \tilde{i}(\varphi_H^t(\tilde{\epsilon}(x))), \forall x \in A^*G\} \subset T^*G,$$

is a Lagrangian submanifold of T^*G .

Remark 4.7. We must note that the previous discussion is a particular instance of the construction of Lagrangian bisections on symplectic Lie groupoids (see [5]). \diamond

Example 4.8. Let $H : T^*Q \rightarrow \mathbb{R}$ be a Hamiltonian function on T^*Q . In this case, using canonical coordinates (q, p) , the Poisson structure on T^*Q is $\Pi_Q = -\frac{\partial}{\partial q} \wedge \frac{\partial}{\partial p}$ and the Hamiltonian vector field is given by

$$X_H = \frac{\partial H}{\partial p} \frac{\partial}{\partial q} - \frac{\partial H}{\partial q} \frac{\partial}{\partial p}.$$

Now, let $G = Q \times Q$ be the pair groupoid. Then, on $T^*(Q \times Q)$ the source map $\tilde{\alpha} : T^*(Q \times Q) \rightarrow T^*Q$ is just $(q_0, q_1, p_0, p_1) \mapsto (q_0, -p_0)$. As a consequence, since $\Pi_{Q \times Q} = \frac{\partial}{\partial q_0} \wedge \frac{\partial}{\partial p_0} + \frac{\partial}{\partial q_1} \wedge \frac{\partial}{\partial p_1}$, we have

$$\begin{aligned} d\tilde{\alpha}^*H &= \frac{\partial H}{\partial q_0} dq_0 - \frac{\partial H}{\partial p_0} dp_0 \\ X_{\tilde{\alpha}^*H} &= \frac{\partial H}{\partial p_0} \frac{\partial}{\partial q_0} + \frac{\partial H}{\partial q_0} \frac{\partial}{\partial p_0} \end{aligned}$$

Finally, since $\tilde{\epsilon} : A^*G \rightarrow T^*G$, $(q, p) \mapsto (q, q, -p, p)$, we have that

$$S_H^t = \{(q, (\psi_1)_H^t(q, p), -p, (\psi_2)_H^t(q, p)) \mid (q, p) \in T^*Q\},$$

where $(q, p) \mapsto \psi_H^t(q, p) = \left((\psi_1)_H^t(q, p), (\psi_2)_H^t(q, p) \right)$ is the flow of X_H .

Remark 4.9. Let $L : TQ \rightarrow \mathbb{R}$ be a regular continuous Lagrangian on the manifold Q . *The exact discrete Lagrangian for a small time-step $h > 0$* ([17]) is the

discrete Lagrangian $L_d^h : Q \times Q \rightarrow \mathbb{R}$ given by

$$L_d^h(q_0, q_1) = \int_0^h L(\sigma_{01}(t), \dot{\sigma}_{01}(t)) dt$$

where $\sigma_{01}(t)$ is the unique solution of the Euler-Lagrange equations for L which satisfies the boundary conditions $\sigma_{01}(0) = q_0$ and $\sigma_{01}(h) = q_1$.

The Legendre transformations, $\mathbb{F}_{L_d^h}^\pm$ and $\mathbb{F}L$, for L_d^h and L respectively, are related as follows (see [17])

$$\begin{aligned} \mathbb{F}^+ L_d^h(q_0, q_1) &= \mathbb{F}L(\sigma_{01}(h), \dot{\sigma}_{01}(h)) \\ \mathbb{F}^- L_d^h(q_0, q_1) &= \mathbb{F}L(\sigma_{01}(0), \dot{\sigma}_{01}(0)) \end{aligned}$$

Therefore, if L is (hyper-)regular then L_d^h is (hyper-)regular.

Suppose that L is hyperregular. Denote by $H : T^*Q \rightarrow \mathbb{R}$ the corresponding Hamiltonian function, i.e.

$$H = E_L \circ \mathbb{F}L^{-1}$$

E_L being the Lagrangian energy associated with L .

In [17] it is proved that for a small time-step h , the following diagrams relate the flow $\psi_H^h : T^*Q \rightarrow T^*Q$ at time h of the Hamiltonian vector field $X_H \in \mathfrak{X}(T^*Q)$ and the discrete Lagrangian evolution operator $\Upsilon_{L_d^h} : Q \times Q \rightarrow Q \times Q$ associated with L_d^h defined as in (4.2)

$$\begin{array}{ccc} Q \times Q & \xrightarrow{\Upsilon_{L_d^h}} & Q \times Q \\ \mathbb{F}_{L_d^h}^+ \downarrow & & \downarrow \mathbb{F}_{L_d^h}^+ \\ T^*Q & \xrightarrow{\psi_H^t} & T^*Q \end{array} \quad \begin{array}{ccc} Q \times Q & \xrightarrow{\Upsilon_{L_d^h}} & Q \times Q \\ \mathbb{F}_{L_d^h}^- \downarrow & & \downarrow \mathbb{F}_{L_d^h}^- \\ T^*Q & \xrightarrow{\psi_H^t} & T^*Q \end{array}$$

A direct computation, using this result, shows that

$$\Psi(S_{L_d^h}) = (\Psi \circ dL_d^h)(Q \times Q) = \text{graph } \psi_H^h = \Psi(S_H^h)$$

Here $\Psi : T^*(Q \times Q) \rightarrow T^*(Q \times Q)$ is the isomorphism described in (4.1). Thus, $S_{L_d^h} = S_H^h$.

It would be interesting to extend this result for a general regular Lagrangian $L : AG \rightarrow \mathbb{R}$ on the associated Lie algebroid AG of a Lie groupoid $G \rightrightarrows M$. Previously, it is necessary to describe the exact discrete Lagrangian $L_d^h : G \rightarrow \mathbb{R}$ induced by L (see [19] for more details).

The results of this section motivate that we may introduce, in the general context of a Lie groupoid G , the notion of an implicit discrete Hamiltonian system as a Lagrangian submanifold of T^*G .

◇

5. CONCLUSION

We have developed an algorithm which permits to extract the integrable part of an implicit difference equation. This algorithm produces a sequence of submanifolds (or generally subsets) which encodes where there exist well defined solutions of the discrete dynamics. As an application, our method allows us to easily analyze the case of discrete (nonholonomic) Lagrangian systems from an implicit point of view, in particular the situation of singular lagrangians, and we will expect that it will be an useful tool for discrete optimal control theory. We will discuss this topic in a future paper.

Finally, in our opinion, the developments about implicit systems defined by Hamiltonian functions open doors to future research in the theory of generating functions for Poisson morphisms (using standard symplectic techniques), the theory of discrete hamiltonian systems in a Poisson context [1], the Dirac theory of constraints in the discrete setting, a discrete Hamilton-Jacobi theory... among other research topics.

APPENDIX A. DISCRETE LAGRANGIAN MECHANICS ON LIE GROUPOIDS

A **discrete Lagrangian system** consists of a Lie groupoid $G \rightrightarrows M$ (the **discrete space**) and a **discrete Lagrangian** $L : G \rightarrow \mathbb{R}$.

For $g \in G$ fixed, we consider the set of **admissible sequences**:

$$\mathcal{C}_g^N = \{ \gamma_G : [1, N] \cap \mathbb{Z} \rightarrow G \mid \gamma_G \text{ is an admissible sequence and } \gamma_G(1) \cdot \gamma_G(N) = g \}.$$

We may identify the tangent space to \mathcal{C}_g^N at γ_G with

$$T_{\gamma_G} \mathcal{C}_g^N \equiv \{ (v_1, v_2, \dots, v_{N-1}) \mid v_k \in A_{x_k} G \text{ and } x_k = \beta(\gamma_G(k)), 1 \leq k \leq N-1 \}.$$

$(v_1, v_2, \dots, v_{N-1})$ is called an *infinitesimal variation* of γ_G .

Now, we define the **discrete action sum** associated to the discrete Lagrangian $L : G \rightarrow \mathbb{R}$ by

$$\mathcal{S}L(\gamma_G) = \sum_{k=1}^N L(\gamma_G(k)).$$

Hamilton's principle requires that this discrete action sum be stationary with respect to all the infinitesimal variations. This requirement gives the following alternative expressions for the **discrete Euler-Lagrange equations** (see [18]):

$$\overleftarrow{X}(g_k)(L) - \overrightarrow{X}(g_{k+1})(L) = 0, \quad (\text{A.1})$$

for all sections X of $\tau : AG \rightarrow M$. Alternatively, we may rewrite the discrete Euler-Lagrange equations as

$$d[L \circ l_{g_k} + L \circ r_{g_{k+1}} \circ i](\epsilon(x_k))|_{A_{x_k} G} = 0,$$

where $\beta(g_k) = \alpha(g_{k+1}) = x_k$.

Thus, we may define the **discrete Euler-Lagrange operator** $D_{\text{DEL}}L : G_2 \rightarrow A^*G$ from G_2 to A^*G , the dual of AG . This operator is given by

$$D_{\text{DEL}}L(g, h) = d[L \circ l_g + L \circ r_h \circ i](\epsilon(x))|_{A_x G}$$

with $\beta(g) = \alpha(h) = x$.

Let $\Upsilon : G \rightarrow G$ be a smooth map such that:

- $\text{graph}(\Upsilon) \subseteq G_2$, that is, $(g, \Upsilon(g)) \in G_2$, for all $g \in G$ (Υ is a **second order operator**).
- $(g, \Upsilon(g))$ is a solution of the discrete Euler-Lagrange equations, for all $g \in G$, that is, $(D_{\text{DEL}}L)(g, \Upsilon(g)) = 0$, for all $g \in G$.

In such a case

$$\overleftarrow{X}(g)(L) - \overrightarrow{X}(\Upsilon(g))(L) = 0, \quad (\text{A.2})$$

for every section X of AG and every $g \in G$. The map $\Upsilon : G \rightarrow G$ is called a **discrete flow** or a **discrete Lagrangian evolution operator** for L .

Given a Lagrangian $L : G \rightarrow \mathbb{R}$ we define the **discrete Legendre transformations** $\mathbb{F}^-L : G \rightarrow A^*G$ and $\mathbb{F}^+L : G \rightarrow A^*G$ by

$$\begin{aligned} (\mathbb{F}^-L)(h)(v_{\epsilon(\alpha(h))}) &= -v_{\epsilon(\alpha(h))}(L \circ r_h \circ i) \\ &= \tilde{\alpha}(dL(h))(v_{\epsilon(\alpha(h))}), \quad \text{for } v_{\epsilon(\alpha(h))} \in A_{\alpha(h)}G, \\ (\mathbb{F}^+L)(g)(v_{\epsilon(\beta(g))}) &= v_{\epsilon(\beta(g))}(L \circ l_g) \\ &= \tilde{\beta}(dL(g))(v_{\epsilon(\beta(g))}), \quad \text{for } v_{\epsilon(\beta(g))} \in A_{\beta(g)}G. \end{aligned} \quad (\text{A.3})$$

A discrete Lagrangian $L : G \rightarrow \mathbb{R}$ is said to be **regular** if and only if the Legendre transformation \mathbb{F}^+L is a local diffeomorphism (equivalently, if and only if the Legendre transformation \mathbb{F}^-L is a local diffeomorphism). In this case, if $(g_0, h_0) \in G \times G$ is a solution of the discrete Euler-Lagrange equations for L then, one may prove (see [18]) that there exist two open subsets U_0 and V_0 of G , with $g_0 \in U_0$ and $h_0 \in V_0$, and there exists a (local) discrete Lagrangian evolution operator $\Upsilon_L : U_0 \rightarrow V_0$ such that:

- (i) $\Upsilon_L(g_0) = h_0$,
- (ii) Υ_L is a diffeomorphism and
- (iii) Υ_L is unique, that is, if U'_0 is an open subset of G , with $g_0 \in U'_0$, and $\Upsilon'_L : U'_0 \rightarrow G$ is a (local) discrete Lagrangian evolution operator then

$$\Upsilon_L|_{U_0 \cap U'_0} = \Upsilon'_L|_{U_0 \cap U'_0}.$$

Moreover, if \mathbb{F}^+L and \mathbb{F}^-L are global diffeomorphisms (that is, L is **hyperregular**) then $\Upsilon_L = (\mathbb{F}^-L)^{-1} \circ \mathbb{F}^+L$.

If $L : G \rightarrow \mathbb{R}$ is a hyperregular Lagrangian function, then pushing forward to A^*G with the discrete Legendre transformations, we obtain the **discrete Hamiltonian evolution operator**, $\tilde{\Upsilon}_L : A^*G \rightarrow A^*G$ which is given by

$$\tilde{\Upsilon}_L = \mathbb{F}^\pm L \circ \Upsilon_L \circ (\mathbb{F}^\pm L)^{-1} = \mathbb{F}^+L \circ (\mathbb{F}^-L)^{-1}. \quad (\text{A.4})$$

APPENDIX B. DISCRETE NONHOLONOMIC LAGRANGIAN SYSTEMS

A discrete nonholonomic Lagrangian system on a Lie groupoid $G \rightrightarrows M$ is a **Lagrangian discrete** $L : G \rightarrow \mathbb{R}$, a vector subbundle \mathcal{D}_c (**the constraint distribution**) of the Lie algebroid AG of G and a **discrete constraint embedded submanifold** \mathcal{M}_c of G such that $\dim \mathcal{M}_c = \dim \mathcal{D}_c$.

Let $(L, \mathcal{M}_c, \mathcal{D}_c)$ be a discrete nonholonomic Lagrangian system. The **discrete nonholonomic Euler-Lagrange equations for the system** $(L_d, \mathcal{M}_c, \mathcal{D}_c)$ are given by

$$d(L_d \circ l_g + L_d \circ r_h \circ i)(\epsilon(x))|_{(\mathcal{D}_c)_x} = 0,$$

for $(g, h) \in G_2 \cap (\mathcal{M}_c \times \mathcal{M}_c)$, with $\beta(g) = \alpha(h) = x$ (for more details on this section, see [10]).

For a discrete nonholonomic Lagrangian system, we can define the **discrete nonholonomic Legendre transformations**

$$\mathbb{F}^-(L, \mathcal{M}_c, \mathcal{D}_c) : \mathcal{M}_c \rightarrow \mathcal{D}_c^* \quad \text{and} \quad \mathbb{F}^+(L, \mathcal{M}_c, \mathcal{D}_c) : \mathcal{M}_c \rightarrow \mathcal{D}_c^*$$

as follows:

$$\begin{aligned} \mathbb{F}^-(L, \mathcal{M}_c, \mathcal{D}_c)(h)(v_{\epsilon(\alpha(h))}) &= -v_{\epsilon(\alpha(h))}(L \circ r_h \circ i), \\ &= \tilde{\alpha}(dL(h))(v_{\epsilon(\alpha(h))}), \text{ for } v_{\epsilon(\alpha(h))} \in \mathcal{D}_c(\alpha(h)), \end{aligned} \quad (\text{B.1})$$

$$\begin{aligned} \mathbb{F}^+(L, \mathcal{M}_c, \mathcal{D}_c)(g)(v_{\epsilon(\beta(g))}) &= v_{\epsilon(\beta(g))}(L \circ l_g), \\ &= \tilde{\beta}(dL(g))(v_{\epsilon(\beta(g))}), \text{ for } v_{\epsilon(\beta(g))} \in \mathcal{D}_c(\beta(g)). \end{aligned} \quad (\text{B.2})$$

If $\mathbb{F}^-L_d : G \rightarrow AG^*$ and $\mathbb{F}^+L_d : G \rightarrow AG^*$ are the standard discrete Legendre transformations associated with the Lagrangian function L defined in Equation (A.3) and $i_{\mathcal{D}_c}^* : AG^* \rightarrow \mathcal{D}_c^*$ is the dual map of the canonical inclusion $i_{\mathcal{D}_c} : \mathcal{D}_c \rightarrow AG$ then

$$\mathbb{F}^-(L_d, \mathcal{M}_c, \mathcal{D}_c) = i_{\mathcal{D}_c}^* \circ \mathbb{F}^-L_d \circ i_{\mathcal{M}_c}, \quad \mathbb{F}^+(L_d, \mathcal{M}_c, \mathcal{D}_c) = i_{\mathcal{D}_c}^* \circ \mathbb{F}^+L_d \circ i_{\mathcal{M}_c}. \quad (\text{B.3})$$

Using Equations (B.1) and (B.2), the discrete nonholonomic Euler-Lagrange equations are equivalent to

$$\mathbb{F}^-(L_d, \mathcal{M}_c, \mathcal{D}_c)(h) = \mathbb{F}^+(L_d, \mathcal{M}_c, \mathcal{D}_c)(g). \quad (\text{B.4})$$

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